بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

বিস্‌মিল্লাহির রাহ্‌মানির রাহীম

উদ্ভাস

একাডেমিক এন্ড এডমিশন কেয়ার

Class 11: Higher Math (Chapter-1)

MATRICES AND DETERMINANTS

Lecture HM-02

DETERMINANT:

$m \times m$

* The determinant is a scaler value computed from the elements of a square matrix.

rows = column

$|-5| = 5$

$|5| = 5$

$$A = \begin{bmatrix} a_1 & a_2 \\ b_2 & b_2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

2×2 $\quad$ 2×2

$$\begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 8 - 6 = 2$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

3×3 $\quad$ 3×3

Poll Question-01

Which matrix can not refer to a determinant?

(a) Scaler matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) Identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Row matrix

Row = 1

(d) Symmetric matrix

$A^T = A$

MINOR OF DETERMINANT:

The (i,j)-th minor of a m*m determinant is the (m-1)*(m-1) determinant formed by the rest of the entries after deleting i-th row and j-th column (which includes the entry a_{ij}) from the larger m*m determinant.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 2 & 4 & 5 \end{vmatrix}$$

a_{ij} i-th row, j-th col.

1 → minor (1,1) M_{11}

$M_{11} = \begin{vmatrix} 6 & 8 \\ 4 & 5 \end{vmatrix} = 30 - 32 = -2$

$m \times m \to (m-1) \times (m-1)$

4 → (2,1) $M_{2,1} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2$

COFACTOR OF DETERMINANT:

Cofactor is the minor with appropriate sign (+ve or -ve).

cofactor = $\begin{matrix}+\\-\end{matrix}$. minor

The sign in case of (i,j)-th minor is $(-1)^{i+j}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 2 & 4 & 5 \end{vmatrix}$$

$$A_{11} = \underbrace{(-1)^{1+1}}_{\text{sign}} \cdot \underbrace{M_{11}}_{\text{minor}} = \underbrace{(-1)^2}_{\text{sign}} \cdot \underbrace{(-2)}_{\text{minor}} = -2$$

$$A_{32} = \underbrace{(-1)^{3+2}}_{\text{sign}} \cdot \underbrace{\begin{vmatrix} 1 & 3 \\ 4 & 8 \end{vmatrix}}_{\text{minor}} = (-1)^5 \cdot (8-12) = (-1)(-4) = 4$$

i j

Signs for different positions:

1,1 1,2 1,3

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

3,2

3×3

$(-1)^{i+j}$

i,j

$(-1)^{3+2} = (-1)^5 = -1$

Poll Question-02

What's the (2,3)-th cofactor of $\begin{vmatrix} 2 & 5 & 3 \\ 2 & 25 & 4 \\ 4 & 12 & 6 \end{vmatrix}$?

i j

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

(a) 4 (b) -2 (c) 6 (d) -4

$$-\begin{vmatrix} 2 & 5 \\ 4 & 12 \end{vmatrix} = -(24-20) = -4$$

VALUE OF DETERMINANT:

A determinant can be expanded with any rows/columns while calculating the value.

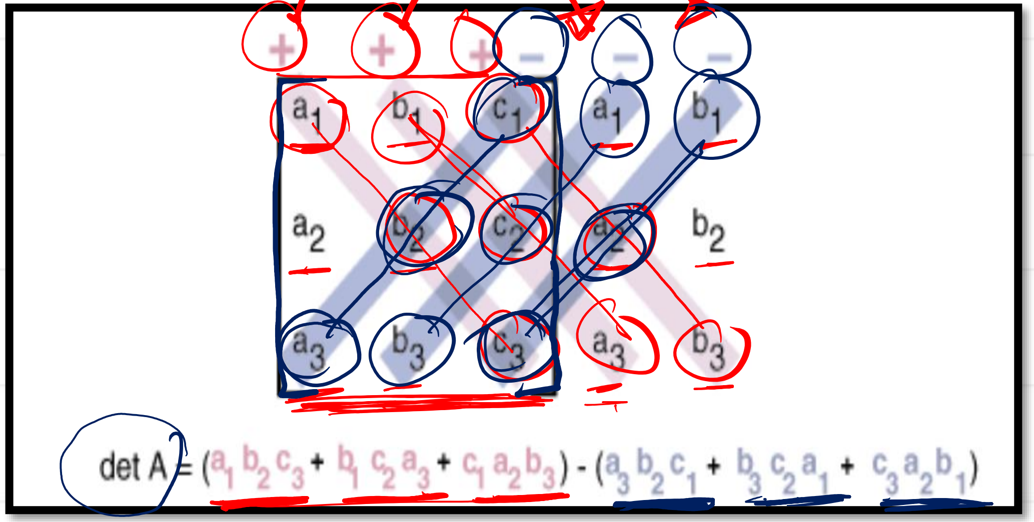
$$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

$$= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 5 \end{vmatrix} \begin{matrix} 1 & 2 \\ 0 & 2 \\ 1 & 0 \end{matrix}$$

$$(10+0+0) - (0+0+0) = 10$$



$A \Rightarrow |A| = \det A$

$+$ $-$

VALUE OF DETERMINANT:

A_{11}

A determinant can be expanded with any rows/columns while calculating the value.

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \cdot A_1 + b_1 \cdot B_1 + c_1 \cdot C_1$$

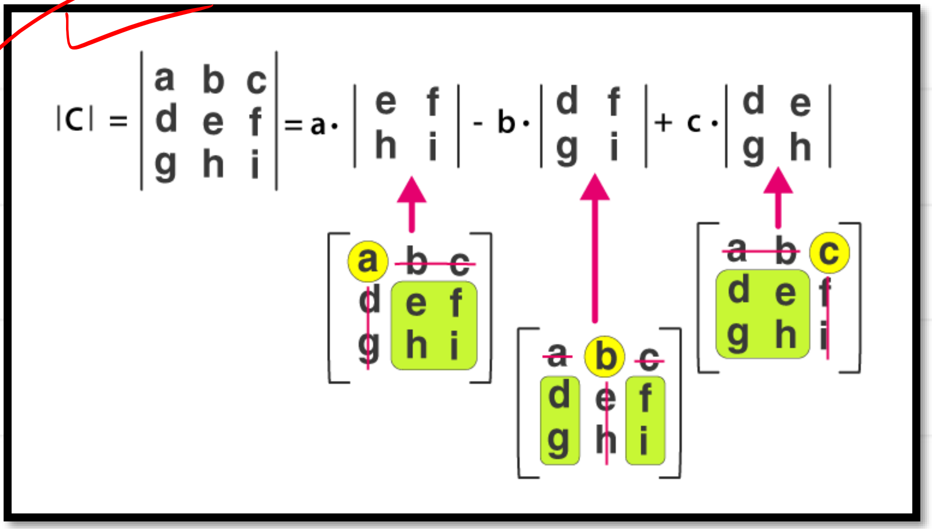
$$= a_2 \cdot A_2 + b_2 \cdot B_2 + c_2 C_2$$

$$a_1\left(+\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right) + b_1\left(-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}\right) + c_1\left(+\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$ {1ST ROW}

$$a_1\left(+\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right) + a_2\left(-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}\right) + a_3\left(+\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right)$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$ {1ST COLUMN}



$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Poll Question-03

What’s the value of $\begin{vmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{vmatrix}$?

(a) 4 (b) 8 (c) 16 (d) 0

$$2 \cdot \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} - 0(\quad) + 0(\quad)$$

$$2 \cdot (2\times 2 - \underbrace{2\times 0}_{0}) - 0 + 0 = 2\times 2\times 2 = 8$$

ACTIVITY:

Calculate the determinants:

1(i). $\begin{vmatrix} 16 & 5 & 6 \\ 12 & 4 & 7 \\ 17 & 6 & 10 \end{vmatrix}$

1(ii). $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$

PROPERTIES OF DETERMINANT:

1. The value of determinant remains unaltered if its rows are changed into columns and the columns into rows.

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Rows ⟷ column

column ⟷ Row

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

b_3c_2

c_2b_3

PROPERTIES OF DETERMINANT:

2. The interchange of any two rows or two columns of the determinant changes its sign.

$+ \rightarrow -$

$- \rightarrow +$

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

r_1, r_2 interchange

$$|A| = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= -\left(a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right)$$

$(-1)^{2+1} = -$

PROPERTIES OF DETERMINANT:

3. If all the elements of a row or column are identical/same to the elements of some other row or column, then the determinant is zero (0).

$$|A| = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = D$$

expand → 0

$$|A|_{exchange} = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = -D$$

$\boxed{\text{1st col} \Leftrightarrow \text{2nd Col}}$

$$|A| = |A|_{exchange}$$

$$D = -D$$

$$\Rightarrow 2D = 0$$

$$\Rightarrow \boxed{D = 0}$$

PROPERTIES OF DETERMINANT:

4. If all the elements of a row or a column are multiplied by a non-zero constant, then the value of determinant gets multiplied by the same constant.

$$|A| = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A_1 \quad B_1 \quad C_1$$

$$L.H.S = ma_1 A_1 + mb_1 B_1 + mc_1 C_1 = m(a_1A_1 + b_1B_1 + c_1C_1)$$

$$= m \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = R.H.S$$

PROPERTIES OF DETERMINANT:

5. If all the elements of a row or column are proportional to the respective elements of some other row or column, then the determinant is zero (0).

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ ma_1 & mb_1 & mc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

1st row

m× 1st row

$$= m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

same

$$= m \times 0 = 0$$

PROPERTIES OF DETERMINANT:

6. If all the elements of a particular row or a column are the sum of two different entries, then the determinant can be expressed by the sum of two different determinants.

$$|A| = \begin{vmatrix} a_1 + p & b_1 & c_1 \\ a_2 + q & b_2 & c_2 \\ a_3 + r & b_3 & c_3 \end{vmatrix}$$

A_1, A_2, A_3

$$= (a_1+p)A_1 + (a_2+q)A_2 + (a_3+r)A_3$$

$$= (a_1A_1 + a_2A_2 + a_3A_3) + (pA_1 + qA_2 + rA_3)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}$$

EXERCISE 1.2:

18. Prove that $\begin{vmatrix} 1 & x-a & y-b \\ 1 & x_1-a & y_1-b \\ 1 & x_2-a & y_2-b \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$

$$L.H.S = \begin{vmatrix} 1 & x-a & y \\ 1 & x_1-a & y_1 \\ 1 & x_2-a & y_2 \end{vmatrix} - \begin{vmatrix} 1 & x-a & b \\ 1 & x_1-a & b \\ 1 & x_2-a & b \end{vmatrix}$$

$\begin{matrix} 1 & 1 & 1 \end{matrix} \times b$

$\begin{matrix} b & b & b \end{matrix}$

proportional

0

$$= \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} - \begin{vmatrix} 1 & a & y \\ 1 & a & y_1 \\ 1 & a & y_2 \end{vmatrix}$$

0

$\begin{matrix} 1 & 1 & 1 \end{matrix} \times a$

$\begin{matrix} a & a & a \end{matrix}$

$\times a$

1st col → 2nd col ⇒ proportional

$= R.H.S$

PROPERTIES OF DETERMINANT:

7. If all the elements of a row or column are multiplied by same constant and then added to or subtracted from the respective elements of some other row or column, the value of determinant remains unchanged.

$$|A| = \begin{vmatrix} a_1 \pm mb_1 & b_1 & c_1 \\ a_2 \pm mb_2 & b_2 & c_2 \\ a_3 \pm mb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$C_1' = C_1 \pm mC_2$$

$$\boxed{C_1 \pm mC_2}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \pm \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}$$

proportional $= 0$

EXERCISE 1.2:

5. Prove that, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & p & p^2 \\ 1 & p^2 & p^4 \end{vmatrix} = p(p-1)^2(p^2-1)$

$$L.H.S = \begin{vmatrix} 1 & 0 & 0 \\ 1 & p-1 & p^2-p \\ 1 & p^2-1 & p^4-p^2 \end{vmatrix} \qquad \begin{bmatrix} C_3' = C_3 - C_2 \\ C_2' = C_2 - C_1 \end{bmatrix}$$

$$= \begin{vmatrix} p-1 & p(p-1) \\ p^2-1 & p^2(p^2-1) \end{vmatrix} = (p-1)(p^2-1)\begin{vmatrix} 1 & p \\ 1 & p^2 \end{vmatrix}$$

$$= p(p-1)(p^2-1)\begin{vmatrix} 1 & 1 \\ 1 & p \end{vmatrix} = p(p-1)(p^2-1) = R.H.S.$$

9. Prove that, $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (ax^2+2bxy+cy^2)(b^2-ac)$

$r_3' = r_3 - (xr_1 + yr_2)$

$0,\ 0,\ p$

$$LHS = \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ 0 & 0 & -(ax^2+2bxy+cy^2) \end{vmatrix}$$

$$= -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b \\ b & c \end{vmatrix} = -(ax^2+2bxy+cy^2)(ac-b^2) = R.H.S.$$

(b^2-ac)

ACTIVITY:

Prove that,

6(b). $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\,(c-a)$$

4. $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

10. $$\begin{vmatrix} b^2+a^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

19. $$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3-1 & y^3-1 & z^3-1 \end{vmatrix} = (xyz-1)(x-1)(y-1)(z-1)$$

20. $$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Poll Question-04

What's the value of $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$?

(a) $4(a-b)(b-c)(c-a)$

(b) $(a-b)(b-c)(c-a)$

(c) $abc(a-b)(b-c)(c-a)$

(d) $4(a+b)(b+c)(c+a)$

$abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= abc\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a+b)(a-b) & (b+c)(b-c) & c^2 \end{vmatrix}$

$C_1' = C_1 - C_2$

$C_2' = C_2 - C_3$

$= abc\begin{vmatrix} a-b & b-c \\ (a+b)(a-b) & (b+c)(b-c) \end{vmatrix}$

$= abc(a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$

PRACTICE RPOBLEM

15. Prove that, $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

লেগে থাকো সৎভাবে,

স্বপ্ন জয় তোমারই হবে

উদ্ভাস-উন্মেষ শিক্ষা পরিবার

THANK YOU