بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

বিস্‌মিল্লাহির রাহ্‌মানির রাহীম

উদ্ভাস

একাডেমিক এন্ড এডমিশন কেয়ার

Class Twelve: H.Math (Chapter-05)

Binomial theorem

Lecture: HM-07

Binomial Expression

Binomial Expression:

A polynomial equation with two terms usually joined by a plus or minus sign is called a binomial expression:

Example: $x + y, 3x + 2, ax + \frac{c}{dy}$

In this case, multiplication/division of two terms is not a binomial

Example: $\frac{x}{y}, xy, \underline{ax^3}$

$\underline{x+y}$ $\quad$ $x-y$ $\quad$ $ax+b$

Binomial Expression

♦ **Discussion about terms in binomial expansion:**

- In the expansion of $(a + x)^n$
  If n is non-negative whole number then number of terms will be $(n + 1)$
  If n is negative or fractional then term number of terms will be ∞ [Here considering the conditions $|x| < a$ or $|x| > a$ the expansion is to be done]
- $(a + b \dots \dots .. \text{upto } r\ number)^n$ then number of terms ${}^{n+r-1}C_{r-1}$
- In expansion of $(a + x)^n$, $(r + 1)$th term, $T_{r+1} = {}^nC_r\, a^{n-r}.\, x^r$
- **Middle term:** In case of $(a + x)^n$, if n is even, middle term is $\left(\frac{n}{2} + 1\right)$ th term and if n is odd, middle terms are $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+1}{2} + 1\right)$ th
- If ${}^nC_x = {}^nC_y$ and $x \neq y$, then $n = x + y$.
- In expansion of $(a + x)^n$, ratio of $(r + 1)$ th term and r- th term, $\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}.\frac{x}{a}$

Binomial Expression

- x+y
- $x - y$
- $ax + y$
- $cx - dy$

Binomial Expression

- $x^2 + y^2$
- $x - y^3$
- $ax^2 + y^3$
- $cx^4 - dy^2$

$$\left(\boxed{??} + \boxed{}\right)^{100}$$

$$\boxed{??} - \boxed{}$$

Combination

$$^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

$^{10}C_6 \neq {}^{10}C_4$

a, b, c, d, e

a, b, c

a, b, e

d, a, b

$$^{n}C_{x} = {}^{n}C_{y} \rightarrow n = x + y$$

$^{n}C_x = {}^{n}C_y$

$^{5}C_3$

$n = x + y$

$\rightarrow {}^{n}C_x = {}^{n}C_y$

$n \rightarrow r$

$= {}^{n}C_r = \frac{n!}{(n-r)!\,r!}$

Poll Question 01

- **Which one is not a binomial expression?**

(i) $a + bx$ (ii) $b + \frac{8}{3y}$ (iii) $ab + xy + 2$ (iv) $ax^2 + \frac{cy^2}{dz^{-3}}$

দ্বিপদী নয়

Binomial Expression

গাণিতিক আরোহ বিধিঃ

1. আরোহ বিধি ও আরোহ পদ্ধতি:

গাণিতিক আরোহ বিধি (Principle of Mathematical Induction) ঃ যদি $\mathbb{N}$ স্বার্ভাবিক সংখ্যার সেট এবং অপর একটি সেট $S \subset \mathbb{N}$ এরূপ হয় যে $1 \in S$ এবং (ii) $m \in S$ হলে, $m + 1 \in S$ (যেখানে $m \in \mathbb{N}$) । তাহলে, $S = \mathbb{N}$, এর একটি মৌলিক স্বীকার্য। এ স্বীকার্যকে গাণিতিক আরোহ বিধি বলা হয়।

গাণিতিক আরোহ পদ্ধতিঃ চলরাশি স্বাভাবিক সংখ্যা $n \in \mathbb{N}$ সম্বলিত কোন উক্তি যদি $n = 1$ এর জন্য সত্য হয় এবং উক্তিটি $n = m \in \mathbb{N}$ এর জন্য সত্য ধরে যদি তা $n = m + 1 \in \mathbb{N}$ এর জন্যও সত্য হয়, তবে উক্তিটি সকল $n \in \mathbb{N}$

If $\mathbb{N}$ is set of natural numbers ans ssn is such a set that, if (i) $1 \in S$ and (ii) $m \in S$ is true then $m + 1 \in S$, then $S = \mathbb{N}$ This is a fundamental postulate of $\mathbb{N}$. This relation is called principal of mathematical induction.

For $n \in \mathbb{N}$, If a statement is true for
(i) $n = 1$
(ii) $m = 1 \in \mathbb{N}$ after assuming that the statement is true $n = m \in \mathbb{N}$
Then the statement is true for all $n \in \mathbb{N}$

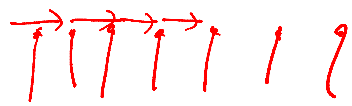
Binomial Expression

Principle of Mathematical Induction

$n = 1$ True?

$n = m$ True $\rightarrow$ $(m+1)$ True

$n = \boxed{} \rightarrow n$



$1, 2, 3, 4, \ldots$

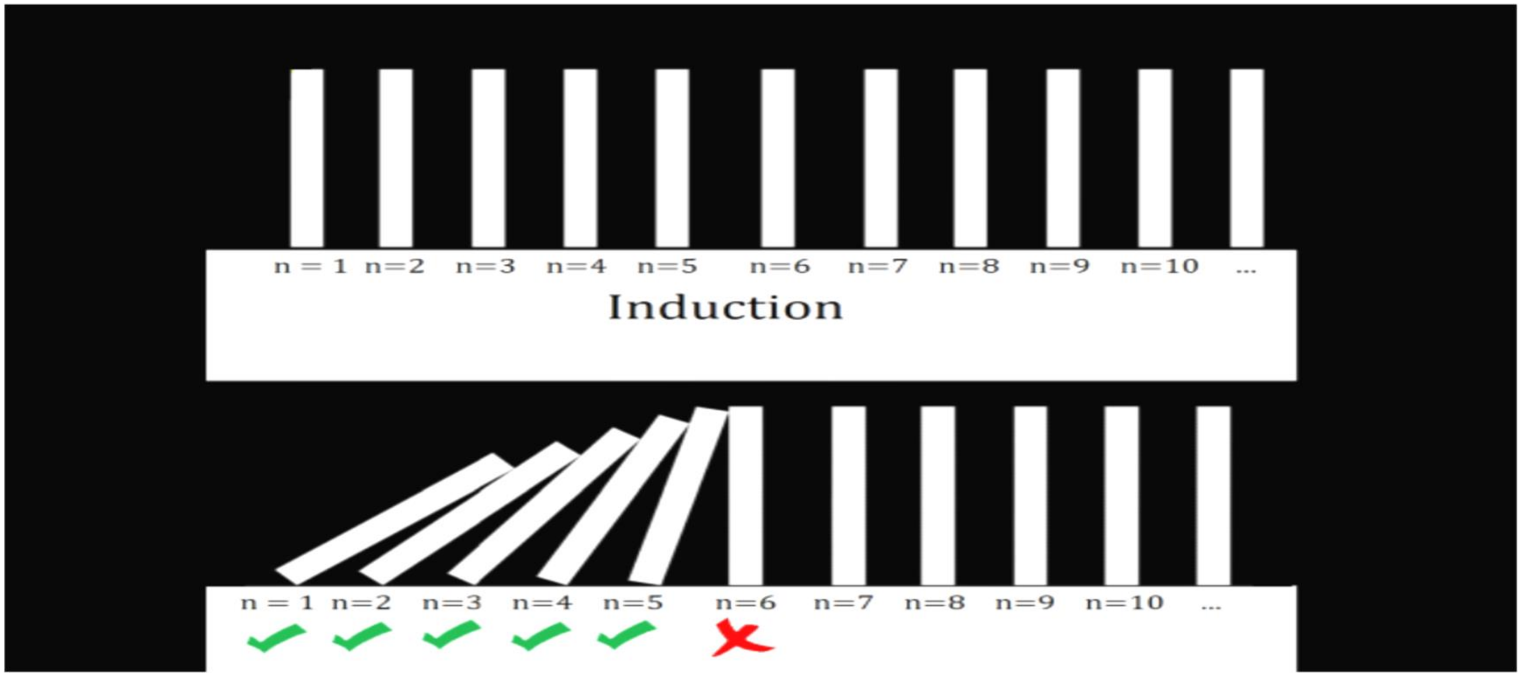
$n = 1$ LHS = RHS

$n = m$

$(m+1)$

Binomial Expression

Principle of Mathematical Induction:

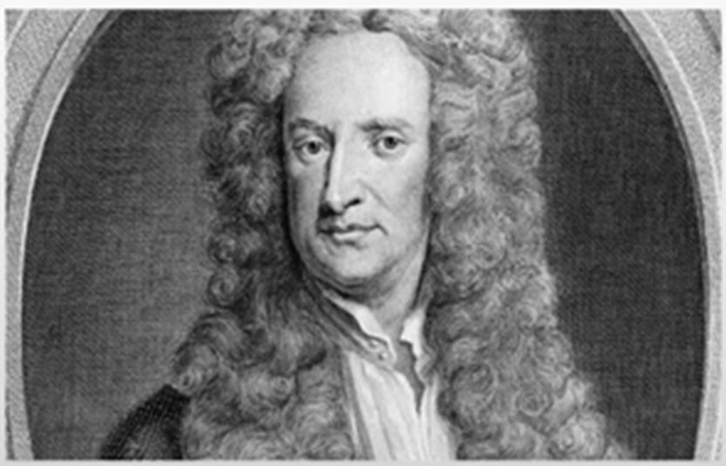


Binomial Expression

Binomial Theorem:

$(a + x)^n = a^n + {}^nC_1\, a^{n-1}\, x + {}^nC_2\, a^{n-2}\, x^2 + {}^nC_3\, a^{n-3}\, x^3 + \ldots \ldots \ldots + {}^nC_r\, a^{n-r}\, x^r + \ldots \ldots \ldots + x^n$

Isaac Newton discovered this theorem about 1665 and later stated, in 1676, with proof, the general form of the theorem (for any real number n), and a proof by john Colson was published in 1736



Sir Isaac Newton

John Colson

Binomial Expression

Example:

$$(a+x)^2 = \underline{a^2 + 2ax + x^2}$$

$$\underline{(a+x)}\,\underline{(a+x)}$$

$$(a+x)^3 = \underline{(a+x)(a+x)}(a+x) = \boxed{}$$

$$(a+x)^{100} = \ldots$$

Binomial Expression

Binomial Theorem:

Proof 1:

proof by induction:

$$\left\{ (a+x)^n = a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + {}^nC_3 a^{n-3} x^3 + \dots \right.$$

$n = 1$ হলে,

$$LHS = (a+x)^n = (a+x)^1 = a+x$$

$$RHS = a^1 + {}^1C_1 a^{1-1} x + \boxed{\cancel{{}^1C_2} \dots} = a + 1 \cdot a^0 \cdot x = a + x$$

Binomial Expression

Binomial Theorem:

$1, 2, 3, 4, \ldots$

Proof 1: $n = m \Rightarrow$ True

$$(a+x)^m = a^m + {}^mC_1 a^{m-1}x + {}^mC_2 a^{m-2}x^2 + {}^mC_3 a^{m-3}x^3 + \ldots$$

$$\Rightarrow (a+x)(a+x)^m = (a+x)\left(a^m + {}^mC_1 a^{m-1}x + {}^mC_2 a^{m-2}x^2 + {}^mC_3 a^{m-3}x^3 + \ldots\right)$$

$$\Rightarrow (a+x)^{m+1} = a^{m+1} + {}^mC_1 a^m x + {}^mC_2 a^{m-1}x^2 + \ldots + a^m x + {}^mC_1 a^{m-1}x^2 + {}^mC_2 a^{m-2}x^3 + \ldots$$

$$= a^{m+1} + ({}^mC_1 + 1)a^m x + ({}^mC_1 + {}^mC_2)a^{m-1}x^2 + ({}^mC_2 + {}^mC_3)a^{m-2}x^3 + \ldots$$

$$= a^{m+1} + (m+1)a^{(m+1)-1}x + {}^{m+1}C_2 a^{(m+1)-2}x^2 + {}^{m+1}C_3 a^{(m+1)-3}x^3 + \ldots$$

$$(a+x)^{m+1} = a^{m+1} + {}^{m+1}C_1 a^{(m+1)-1}x + {}^{m+1}C_2 a^{(m+1)-2}x^2 + \ldots$$

Binomial Expression

Binomial Theorem:

$a^2 \quad x^2$

$(a+x)^2 = (a+x)(a+x)$

ax

ax

Proof 2:

Suppose, we want to determine $(a+x)^2$. Now, $(a+x)^2$ is equivalen to $(a+x)(a+x)$. This multiplication can be done in follow way:

Probable term from first (a+x)	Probable term from first (a+x)	Multiplication	Number of ways to select
a	a	a^2	1
a	x	ax	1
x	a	ax	1
x	x	x^2	1
		$a^2 + 2ax + x^2$	

$2ax$

Binomial Expression

Binomial Theorem:

Proof 2:

$(a+x)^3 = (a+x)(a+x)(a+x)$

a a a $\rightarrow a^3$

a a x $\quad a^2x$

After expanding $(a+x)^3$ if we want a^3, we will need three a's. From three $(a+x)$ we can select thrice a's in ${}^3C_3$ or 1 way.

After expanding $(a+x)^3$ if we want a^2x, we will need two a's. From three $(a+x)$ we can select two a's in ${}^3C_2$ or 3 ways.

After expanding $(a+x)^3$ if we want ax^2, we will need one a. From three $(a+x)$ we can select one a in ${}^3C_1$ or 3 ways.

After expanding $(a+x)^3$ if we want x^3, we will need 0 a. From three $(a+x)$ we can select 0 a in ${}^3C_0$ or 1 ways.

a a x
a x a
x a a

Binomial Expression

Binomial Theorem:

$^nC_x = {^nC_y} \rightarrow n = x + y$

Proof 2:

Therefore,

$$(a+x)^3 = a^3 + {^3C_2}a^2x + {^3C_1}a\,x^2 + x^3,$$

$$\text{Or, } (a+x)^3 = a^3 + {^3C_1}a^2x + {^3C_2}a\,x^2 + x^3 \;(\text{As } {^3C_1} = {^3C_2})$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

Similarly,

$$(a+x)^n = an + {^nC_1}a^{n-1}x + {^nC_2}a^{n-2}x^2 + \dots\dots + x^n$$

$$(a+x)^n = a^n + {^nC_1}a^{n-1}x + {^nC_2}a^{n-2}x^2 + {^nC_3}a^{n-3}x^3 + \dots$$

Poll Question 02

- **How many terms there will be in expansion of** $(a+2x)^{21}$ **?**

(i) 20 (ii) 21 (iii) 22 (iv) 23

$(a+x)^2 \longrightarrow 3$

$(a+x)^3 \longrightarrow 4$

$(a+x)^n \longrightarrow n+1$

Binomial Expression

Pascal's Triangle:

1

1 1

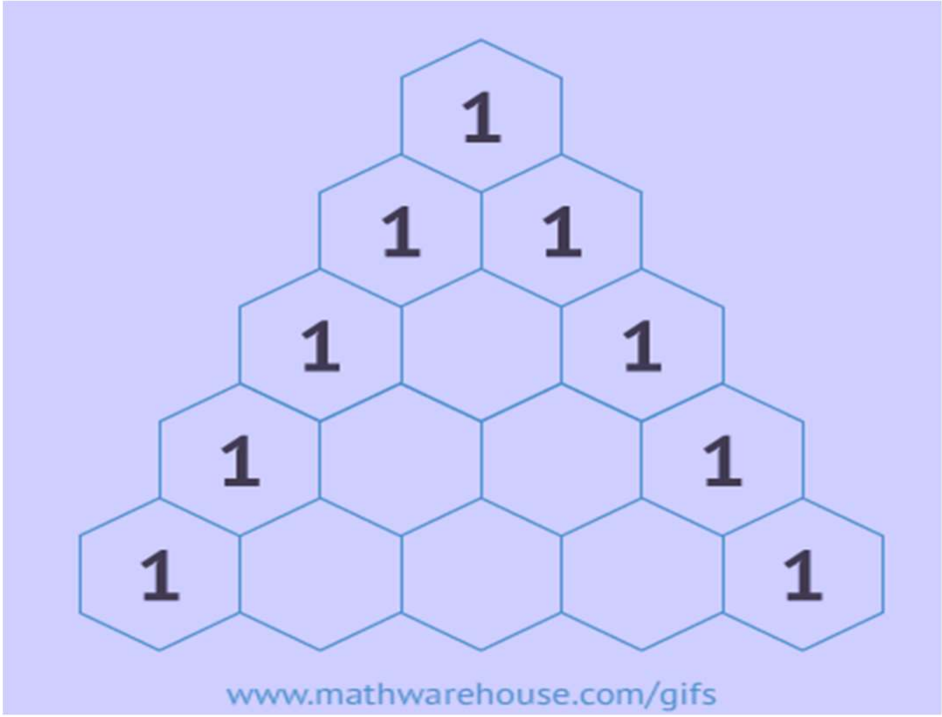
1 2 1

1 3 3 1

1 4 6 4 1

Binomial Expression

Pascal's Triangle:



Binomial Expression

Pascal's Triangle:

Use:

Binomial Expression

Pascal's Triangle:

Use:

1 2 1

$(a + x)^2 = a^2 + 2\,ab + b^2$

$(a + x)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$ 1 3 3 1

$(a + x)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Binomial Expression

(r+1) th term

$$(a + x)^n = a^n + {}^nC_1\, a^{n-1}\, x + {}^nC_2\, a^{n-2}\, x^2 + {}^nC_3\, a^{n-3}\, x^3 + \dots \dots$$

Here,

1st term = a^n

2nd term = ${}^nC_1\, a^{n-1}\, x$

3rd term = ${}^nC_2\, a^{n-2}\, x^2$

4th term = ${}^nC_3\, a^{n-3}\, x^3$

.

.

.

(r+1)-th term = ${}^nC_r\, a^{n-r} x^r$

$T_{10} = {}^nC_9\, a^{n-9}\, x^9$

$$\boxed{T_{r+1} = {}^nC_r\, a^{n-r}\, x^r}$$

$T_{200} = T_{\underbrace{199}_{r}+1} = {}^nC_{199}\, a^{n-199}\, x^{199}$

Binomial Expression

(r+1)-th term

$$T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, x^{r}$$

Binomial Expression

Equidistant terms:

$$(a + x)^n = a^n + {}^nC_1\, a^{n-1}\, x + {}^nC_2\, a^{n-2}\, x^2 + {}^nC_3\, a^{n-3}\, x^3 + \ldots \ldots \ldots + {}^nC_r\, a^{n-r}\, x^r + \ldots \ldots \ldots + x^n$$

If $a = 1$

$$(1 + x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + {}^nC_3\, x^3 + \ldots \ldots \ldots + {}^nC_r\, x^r + \ldots \ldots + {}^nC_{n-2}\, x^{n-2} + {}^nC_{n-1}\, x^{n-1} + x^n$$

By complementary combination,

$${}^nC_1 = {}^nC_{n-1}$$
$${}^nC_2 = {}^nC_{n-2}$$
$$\vdots$$
$${}^nC_r = {}^nC_{n-r}$$

Therefore, The binomial coefficients which are equidistant from the beginning and from the ending are equal (সমান)

Binomial Expression

Middle term:

$$(a+x)^2 = a^2 + 2ab + b^2$$

$$(a+x)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+x)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$n = 9$, 10টি পদ

$\frac{n+1}{2}, \frac{n+1}{2}+1$

5 6

$\frac{n+1}{2}, \frac{n+1}{2}+1$

$(a+x)^n \rightarrow$ পদসংখ্যা $n+1$

3

4, 5

$\left(\frac{4}{2}+1\right)$

5 $\frac{4}{2}$

$\frac{n}{2}+1$

$\frac{4}{2}+1$

Binomial Expression

Middle term:

$(a + x)^2 = a^2 + 2ab + b^2$ $\longrightarrow$ 2

$(a + x)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ $\longrightarrow$ 2 , 3

$(a + x)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ $\longrightarrow$ 3

Binomial Expression

<u>Middle term:</u>

If n is even-

$\frac{n}{2} + 1$

Binomial Expression

<u>Middle term:</u>

If n is even: $\frac{n}{2} + 1$ th term

$\frac{n+1}{2}$

If n is odd: $\frac{n+1}{2}$ and $\frac{n+1}{2} + 1$ th term

$\frac{n+1}{2} + 1$

Ratio of consecutive terms:

$$\frac{T_{r+1}}{T_r} = \frac{{}^{n}C_{r}\, an^{-r}\, xr}{{}^{n}C_{r-1} a^{n-r+1} x^{r-1}}$$

$$= \frac{n-r+1}{r}\,\frac{x}{a}$$

Binomial Expression

Ratio of consecutive terms:

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\, a^{n-r}\, x^r}{{}^nC_{r-1}\, a^{n-r+1} x^{r-1}}$$

$$= \frac{n-r+1}{r}\,\frac{x}{a}$$

$$= \frac{{}^nC_r\ a^{n-r}\ x^r}{{}^nC_{r-1}\ a^{n-r+1}\ x^{r-1}}$$

$$= \frac{\frac{\cancel{n!}}{r!(n-r)!} \times \cancel{a^{n-r}} \times x^r}{\frac{\cancel{n!}}{(r-1)!(n-r+1)!} \times \cancel{a^{n-r+1}} \times x^{r-1}}$$

$$= \frac{(r-1)!(n-r+1)!}{r!\,(n-r)!}\ \frac{1}{a}\cdot\frac{x}{1}$$

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\ a^{n-r}\ x^r}{{}^nC_{r-1}\ a^{n-(r-1)}\ x^{r-1}}$$

Binomial Expression

1. Expand: $(x+ 3y)^4$

$(x+3y)^4$

$= x^4 + 4c_1 x^3 (3y)^1 + 4c_2 x^2 (3y)^2 + 4c_3 x (3y)^3 + 4c_4 x^0 (3y)^4$

Binomial Expression

2. Determine 7-th term from expansion of $(1 - \frac{1}{x})^{10}$

$(1 - \frac{1}{x})^{10}$

$a = 1$

$T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, x^{r}$

$x = (-\frac{1}{x})$

$T_{6+1} = {}^{10}C_{6}\, (1)^{4} \times (-\frac{1}{x})^{6}$

Binomial Expression

3. Find the term independent of x in $(2x^3 - \frac{1}{x})^{12}$

$$T_{r+1} = {}^{12}C_r (2x^3)^{12-r} \left(-\frac{1}{x}\right)^r$$

$$= {}^{12}C_r\, 2^{12-r} (x^3)^{12-r} (-1)^r \left(\frac{1}{x^r}\right)$$

$$= {}^{12}C_r\, 2^{12-r}\, x^{36-3r-r} (-1)^r$$

$$= \underline{{}^{12}C_r\, 2^{12-r} (-1)^r}\, x^{36-4r} \rightarrow 1$$

$$x^{36-4r} = 1 = x^0$$

$$36 - 4r = 0$$

$$\underline{r = 9}$$

$${}^{12}C_r\, 2^{12-r} (-1)^r \boxed{x^{36-4r}}$$

$$T_{10} = {}^{12}C_9\, 2^{12-9} (-1)^9 \cdot \cancel{x}$$

Binomial Expression

4. Find the term independent of x in $\left(x^2 - 2 + \frac{1}{x^2}\right)^{12}$

Binomial Expression

5. Find the value of a, if coefficient of x^3 in expansion of $(a + 2x)^5$ is 320.

Binomial Expression

6. If coefficient of x^5 and x^{15} in expansion of $(2x^2 + \frac{k}{x^3})^{10}$ is equal, find the value of k.

$$T_{r+1} = {}^{10}C_r (2x^2)^{10-r} \left(\frac{k}{x^3}\right)^r$$

$$= {}^{10}C_r\, 2^{10-r}\, x^{20-2r-3r}\, k^r$$

$$= {}^{10}C_r\, 2^{10-r}\, x^{20-5r}\, k^r$$

x^5

$$x^{20-5r} = x^5$$

$$20-5r = 5$$

$$r = 3$$

$$T_4 = {}^{10}C_3\, 2^{10-3}\, x^5\, k^3$$

x^{15}

$$x^{20-5r} = x^{15}$$

$$20-5r = 15$$

$$r = 1$$

$$T_2 = {}^{10}C_1\, 2^9\, x^{15}\, k^1$$

x^5

$$\underline{{}^{10}C_3\, 2^7 k^3} = \underline{{}^{10}C_1\, 2^9 k}$$

Binomial Expression

7. Determine Middle term:

(i) $\left(\frac{x}{y} + \frac{y}{x}\right)^{21}$

$T_{\frac{21+1}{2}}$

$T_{\frac{21+1}{2}+1}$

$T_{11} = {}^{21}C_{10}\left(\frac{x}{y}\right)^{11}\left(\frac{y}{x}\right)^{10}$

$= {}^{21}C_{10}\,\frac{x^{11}}{\cancel{y^{11}}} \times \frac{y^{10}}{\cancel{x^{10}}}$

$= {}^{21}C_{10}\,\frac{x}{y}$

$T_{12} =$

Binomial Expression

8. Determine Middle term:

(ii) $\left(x^2 - 2 + \frac{1}{x^2}\right)^n$

Binomial Expression

9. In expansion of $(a + 3x)^n$, if first three consecutive terms are b , $\frac{21}{2}bx$ and $\frac{189}{4}bx^2$, determine value of a , b and n.

Binomial Expression

10. If n is a natural number, find $(n+1)$ th term from end in expansion of $\left(x^p + \frac{1}{x^p}\right)^n$

Binomial Expression

11. If in expansion of $(1 + x)^n$ ~~sum of odd terms and even terms are~~ consecutively S_1 and S_2, show that $(1 - x^2)^n = S_1^2 - S_2^2$; Where n is a natural number.

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots$$

$$= \underbrace{(1 + {}^nC_2 x^2 + {}^nC_4 x^4 + {}^nC_6 x^6 + \dots)}_{S_1} + \underbrace{({}^nC_1 x + {}^nC_3 x^3 + {}^nC_5 x^5 + \dots)}_{S_2} \quad \dots (1)$$

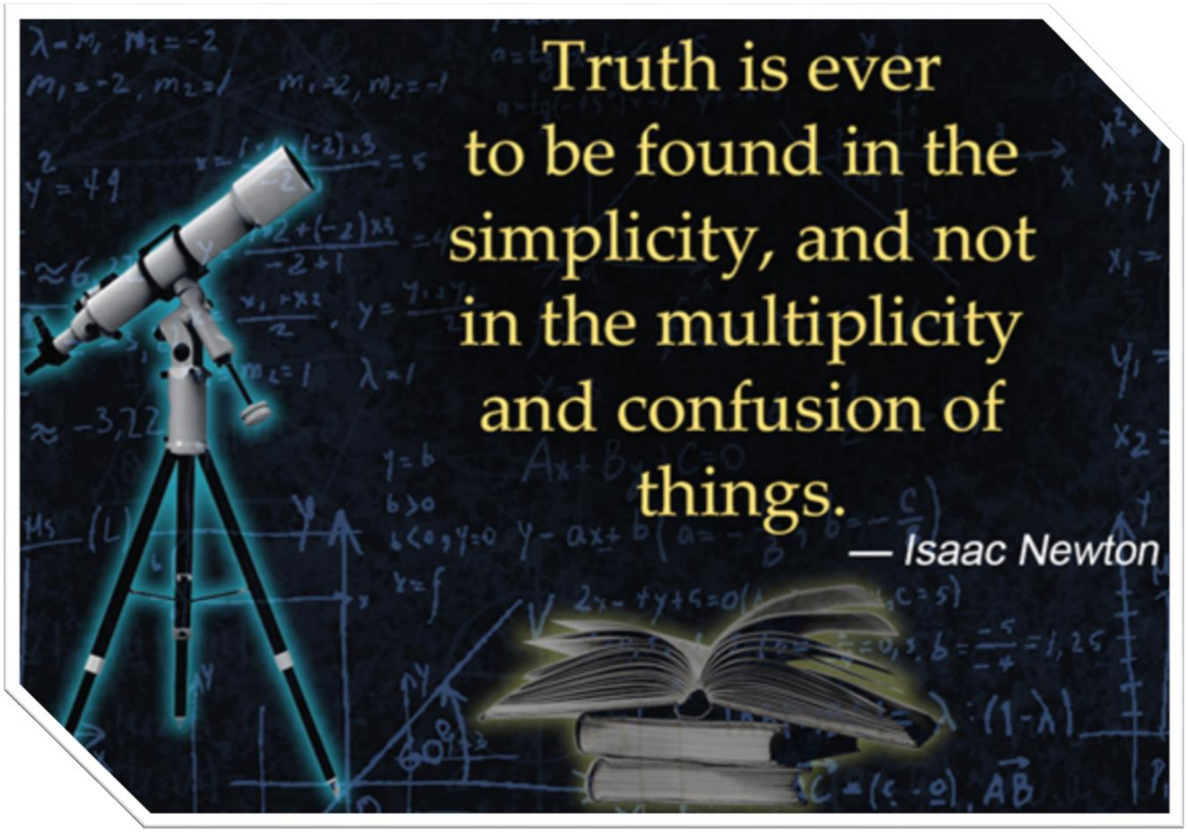
$$= S_1 + S_2$$

$$(1-x)^n = (1 + {}^nC_2 x^2 + {}^nC_4 x^4 + {}^nC_6 x^6 + \dots) + (-{}^nC_1 x - {}^nC_3 x^3 - {}^nC_5 x^5 - \dots)$$

$$= \underbrace{(1 + {}^nC_2 x^2 + {}^nC_4 x^4 + \dots)}_{S_1} - \underbrace{({}^nC_1 x + {}^nC_3 x^3 + {}^nC_5 x^5 + \dots)}_{S_2} \quad \dots (II)$$

$$(I) \times (II) \Rightarrow (1-x^2)^n = (S_1 + S_2)(S_1 - S_2) = S_1^2 - S_2^2$$

Binomial Expression



লেগে থাকো সৎভাবে,
স্বপ্ন জয় তোমারই হবে

উদ্ভাস-উন্মেষ শিক্ষা পরিবার

Thank You